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Asymptotic Normality in a Two-Dimensional Random Walk Model for Cell Motility

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We prove the asymptotic normality of a two-dimensional random walk describing the locomotion of cells on planar surfaces and calculate the asymptotic covariance matrix. The trajectories of the walk are random broken lines covered with constant speed, where the time intervals between turns as well as the turn angles are random and stochastically independent.

KEY WORDS: Two-dimensional random walk; cell motion on surfaces; asymptotic normality; asymptotic covariance.

1. INTRODUCTION

For the description of cell motility on planar surfaces the following random walk model is frequently used[1,2]: The trajectory of the microorganism is assumed to consist of straight-line paths of random lengths. After performing such a linear motion for some time, the cell takes a new direction according to some probability distribution. Experimental studies of these phenomena, e.g., treat Escherichia *coli* bacteria,[3] leukocytes,[4–6] slime mold amebae,[7] 3T3 mouse fibroblasts,[8,9] and mycoplasma.[10]

In this paper a stochastic model of the motion described above is given, the asymptotic normal distribution of the position of the cell in the plane for large time instants is proved, and the asymptotic covariance matrix is derived.

Let us proceed to the mathematical formulation of the model. The lengths of the successive time intervals of linear motion will be denoted by $\xi_1, \xi_2, \ldots$. The changes of direction will be given by orthogonal 2×2 matrices $M_1, M_2, \ldots$, so that the $(i+1)$th direction is $P_i e = M_i \cdots M_1 e$,

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where $e \in \mathbb{R}^2$, $\|e\| = 1$, is the initial direction at time 0, when the call starts from the origin. We assume that the trajectory is covered with constant speed 1.

Let

$$\tau(t) = \max \left\{ n \geqslant 0 \,\middle|\, \sum_{i=1}^{n} \xi_i \leqslant t \right\}$$

Then the position at time t is given by

$$X(t) = \sum_{i=1}^{\tau(t)} \xi_i P_{i-1} e + [t - \tau(t)] \, \xi_{\tau(t)+1} P_{\tau(t)} e \tag{1.1}$$

We assume that $(\xi_n)_{n \geqslant 1}$ and $(M_n)_{n \geqslant 1}$ are independent and identically distributed sequences of nonnegative random variables and random orthogonal 2×2 matrices, respectively. Further, we suppose that

$$E(\xi_1^{3+\varepsilon}) < \infty \qquad \text{for some} \quad \varepsilon > 0 \tag{1.2}$$

and that

$$P(M_1 x = \pm x) < 1 \qquad \text{for all} \quad x \in \mathbb{R}^2 \backslash \{0\} \tag{1.3}$$

By condition (1.3), the random walk is not restricted to a line in the plane and is thus truly two-dimensional.

To calculate the exact distribution of $X(t)$ seems to be very hard; see ref. 11 for the apparently only known spcial case.

From (1.3) it follows that $\mathrm{Id} - E(M_1)$ is invertible (Id is the identity 2×2 matrix). For, if $\mathrm{Id} - E(M_1)$ is not invertible, there is an $x \in \mathbb{R}^2 \backslash \{0\}$ satisfying $x = E(M_1)x$. Let π be the projection on the subspace generated by x. Then $\|\pi M_1 x\| \leqslant \|x\|$ with strict inequality if $M_1 x \neq \pm x$, which occurs with positive probability. But then

$$\|x\| = \|\pi(E(M_1)x)\| = \|E(\pi M_1 x)\| \leqslant E(\|\pi M_1 x\|) < \|x\|$$

a contradiction.

By (1.2) the first two moments $\mu = E(\xi_1)$ and $\mu_2 = E(\xi_1^2)$ of ξ_1 are finite, and by (1.3) we can define

$$N = E(M_1)[\mathrm{Id} - E(M_1)]^{-1}$$

The matrix $N + N'$ together with the moments μ and μ_2 of the step lengths will determine the limiting covariance matrix of $X(t)$. Let us now formulate our result.

Theorem 1. Define the rescaled process by

$$X_n(t) = a_n^{-1/2} X(a_n t)$$

where a_n is a sequence of positive constants tending to infinity. Let (1.2) and (1.3) be satisfied. Then for every $T \geqslant 0$,

$$X_n(T) \xrightarrow{D} N\left(0, \frac{T}{2}\left[\frac{\mu_2}{\mu}\,\mathrm{Id} + \mu(N+N')\right]\right) \quad \text{as} \quad n \to \infty \tag{1.4}$$

Our model resembles that of Nossal and Weiss.[12] The main differences are: (1) We allow arbitrary random orthogonal transformations satisfying (1.3), whereas in ref. 12 only rotations are treated. (2) The sequence of times between turns considered here is quite general [only subject to (1.2)], whereas in ref. 12 only exponential ξ_i are studied. (3) In our model ξ_{i+1} and $P_i e$ are independent, whereas Nossal and Weiss allow the parameter of the (exponential) variable ξ_{i+1} to depend on $P_i e$. It might, however, be possible to treat this case by extensions of the methods applied below to prove (1.4).

At the end of their paper,[12] Nossal and Weiss conjecture the asymptotic normality of $X(T)$, stating that "the asymptotic Gaussian property can probably be proved starting from a form of the central limit theorem for weakly dependent random variables." This is exactly our approach to derive (1.4); we make use of a central limit theorem of Rosén.[13] However, it turns out that the application of this result is much more intricate than might be expected at first glance. In fact, the verification of the conditions of the central limit theorem causes many technical difficulties.

Our theorem also gives an easily evaluated expression for the asymptotic covariance matrix, which certainly is not near at hand.

Condition (1.2) is very important for our proof, because we have to utilize various properties of the finite random sequence $\xi_1, \xi_2,..., \xi_{\tau(t)+1}$ of step lengths, which are not satisfied without assuming (1.2). We strongly call in question the validity of the asymptotic normality of $X_n(T)$ if the step lengths do not possess moments of order $3+\varepsilon$.

We now consider two examples.

Example 1. Let M_1 be a random rotation, i.e.,

$$M_1 = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

for some random angle φ taking values in $[0, 2\pi)$, with distribution not concentrated on $\{0, \pi\}$ to ensure (1.3). The asymptotic covariance matrix of $X_n(T)$ in (1.4) is equal to $T\sigma^2 \operatorname{Id}$, where

$$\sigma^2 = \frac{\mu_2}{2\mu} - \frac{\mu}{2} + \frac{\mu}{2} \frac{1 - E(\sin\varphi)^2 - E(\cos\varphi)^2}{E(\sin\varphi)^2 + [1 - E(\cos\varphi)]^2} \tag{1.5}$$

Thus, the two components of $X_n(T)$ asymptotically become independent and normal with the same variance, given by (1.5). In the very special case when ξ_1 is uniformly distributed on $[0, 2\pi)$, we have $\sigma^2 = \mu_2/2\mu$. If additionally the ξ_i are exponential with mean $1/\lambda$, it follows that $\sigma^2 = 1/\lambda$, in accordance with the result in ref. 11.

Example 2. Now suppose that M_1 is, with probability 1, an improper orthogonal matrix so that

$$M_1 = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$$

with some random φ as above. Then the asymptotic covariance of $X_n(T)$ is given by

$$\begin{aligned} &\frac{T}{2}\left[\frac{\mu_2}{\mu}\operatorname{Id} + \mu(N + N')\right] \\ &\quad = T\left[\left(\frac{\mu_2}{2\mu} - \frac{\mu}{2}\right)\operatorname{Id} + \frac{\mu}{2[1 - E(\sin\varphi)^2 - E(\cos\varphi)^2]}\right. \\ &\qquad \left.\times \begin{pmatrix} E(\sin\varphi)^2 + [1 + E(\cos\varphi)]^2, & 2E(\sin\varphi) \\ 2E(\sin\varphi), & E(\sin\varphi)^2 - [1 + E(\cos\varphi)]^2 \end{pmatrix}\right] \end{aligned} \tag{1.6}$$

In this case the components of the limiting normal distribution are independent if and only if $E(\sin\varphi) = 0$. This holds, for instance, if the distribution of φ is symmetric around π.

2. ANALYSIS

The following notation is used. If $y \in \mathbb{R}^2$, $\|y\|$ is the Euclidean length of y, and if C is a 2×2 matrix, $\|C\| = \sup\{\|Cy\| \mid y \in \mathbb{R}^2, \|y\| = 1\}$. The $\langle\, ,\, \rangle$ is the usual scalar product, $[x]$ is the largest integer $\leqslant x$, and $x^+ = \max(x, 0)$. An empty sum is defined to be 0. Let M be a random matrix with the same distribution as the M_i. Conditional or unconditional expected values of random vectors and random matrices are to be understood componentwise. The simple inequality $\|E(M \mid \mathscr{A})\| \leqslant 2^{1/2} E(\|M\| \mid \mathscr{A})$ will be applied sometimes. K, K_1, K_2,... are used as generic symbols for constants, whose values may change from formula to formula.

It turns out that it is more convenient to work with the jump processes

$$\hat{X}(t)=\sum_{i=1}^{\tau(t)} \xi_i P_{i-1} e, \qquad \hat{X}_n(t)=a_n^{-1/2}\hat{X}(a_n t) \tag{2.1}$$

instead of $X(t)$ and $X_n(t)$. The process $\hat{X}(t)$ is constant in each time interval $[\xi_1+\cdots+\xi_{n-1}, \xi_1+\cdots+\xi_n)$ and equal to $X(t)$ for each t of the form $t=\xi_1+\cdots+\xi_n$ for some n. The limiting distributions of $X_n(T)$ and $\hat{X}_n(T)$ as $n\to\infty$ are identical, because

$$\begin{aligned}\|X_n(T)-\hat{X}_n(T)\| &\leqslant a_n^{-1/2}\xi_{\tau(a_n T)+1}\\ &\to 0 \text{ in probability} \quad \text{as } n\to\infty\end{aligned} \tag{2.2}$$

since $a_n\to\infty$ and $\xi_{\tau(a_n T)+1}$ converges in distribution as $n\to\infty$ (see, e.g., ref. 14, p. 371).

We shall use a central limit theorem for dependent random vectors due to Rosén[13] to prove (1.4). Let

$$\hat{X}_{n,i}=a_n^{-1/2}(\hat{X}(a_n Ti/n)-\hat{X}(a_n T(i-1)/n)) \tag{2.3}$$

$$S_{n,\alpha}=\prod_{i=1}^{[\alpha n]} \hat{X}_{n,i}; \qquad \text{in particular, } S_{n,1}=\hat{X}_n(T) \tag{2.4}$$

A special case of Rosén's result, which is appropriate for the problem considered here, can be formulated as follows.

Theorem 2. Assume that the following conditions are satisfied:

1. There is a constant K such that

$$\limsup_{n\to\infty} E(\|S_{n,\beta}-S_{n,\alpha}\|^2)\leqslant K(\beta-\alpha)$$

for all α, $\beta\geqslant 0$ such that $\alpha\leqslant\beta$.

2. We have

$$\lim_{\Delta\to 0+}\frac{1}{\Delta}\limsup_{n\to\infty} E(\|E(S_{n,\alpha+\Delta}-S_{n,\alpha}\mid S_{n,\alpha})\|)=0$$

for all $\alpha\geqslant 0$.

3. There is a 2×2 matrix C satisfying

$$\lim_{\Delta\to 0+}\frac{1}{\Delta}\limsup_{n\to\infty} E(\|E(S_{n,\alpha+\Delta}-S_{n,a})(S_{n,\alpha+\Delta}-S_{n,\alpha})'\mid S_{n,\alpha})-\Delta C\|)=0$$

for all $\alpha\geqslant 0$.

4. We have

$$\lim_{\Delta \to 0+} \frac{1}{\Delta} \limsup_{n \to \infty} \int_{\{\|S_{n,\alpha+\Delta} - S_{n,\alpha}\| \geqslant \varepsilon\}} S_{n,\alpha+\Delta} - S_{n,\alpha}\|^2 \, dP = 0$$

for all $\alpha \geqslant 0$ and all $\varepsilon > 0$.

Then it follows that

$$S_{n,1} \xrightarrow{D} N(0, C) \qquad \text{as} \quad n \to \infty$$

We shall prove 1–4 in a series of four lemmas, where the ith lemma will imply condition i. The matrix C will be identified as $\frac{1}{2}T[(\mu_2/\mu)\,\mathrm{Id} + \mu(N + N')]$.

Conditions 1 and 4 are needed to ensure that $S_{n,1} = X_n(T)$ is a sum of uniformly small terms, which is clearly indispensable for proving asymptotic normality. However, these terms are not the individual summands $\hat{X}_{n,i}$, but segments $S_{n,\alpha+\Delta} - S_{n,\alpha}$ of the total sum. By 1, these segments must have norms with uniformly small second moments; by 2, they must satisfy some kind of Lindeberg condition.

Uniform negligibility of small segments of the sum is not sufficient for establishing asymptotic normality. One also has to impose conditions on the dependence structure of these segments. Conditions 2 and 3 can be very roughly described as stating that, in some sense,

$$E(S_{n,\alpha+\Delta} - S_{n,\alpha} \mid S_{n,\alpha}) \approx 0$$

$$E((S_{n,\alpha+\Delta} - S_{n,\alpha})(S_{n,\alpha+\Delta} - S_{n,\alpha})' \mid S_{n,\alpha}) \approx \Delta C$$

Thus, $S_{n,\alpha}$ approximately behaves like a diffusion process with drift 0 and constant diffusion matrix C. In Lemmas 2 and 3 we shall show that even for fixed $\Delta > 0$ the lim sup terms in 2 and 3 are equal to 0.

For the proofs we shall need some basic facts about the sequence $(\xi_n)_{n \geqslant 1}$, which are essentially known from renewal theory (see refs. 14 and 15):

(a) For $\alpha > 0$ and $i = -1, 0, 1$, $(\xi^{\alpha}_{\tau(t)+i})_{t \geqslant 0}$ is uniformly integrable if $E(\xi_1^{\alpha+1}) < \infty$. Especially if (1.2) holds, we have $E({}^{2+\varepsilon}_{\tau(t)+i}) \leqslant K < \infty$ for all $t \geqslant 0$.

(b) If $l + 1 < i < j \leqslant k$, $t \geqslant s \geqslant 0$, and $f(\xi_1, \xi_2)$ is integrable, $E(f(\xi_i, \xi_j) \mid \tau(t) = k,\ \tau(s) = l)$ does not depend on the pair (i, j), and $E(f(\xi_{l+1}, \xi_j) \mid \tau(t) = k,\ \tau(s) = l)$ does not depend on j.

(c) For all natural numbers a we have

$$E(\tau(t)^a) \leqslant K(1 + t^a) \qquad \text{for all} \quad t \geqslant 0 \tag{2.5}$$

While (a) is a merely technical result, (b) is very intuitive, because it states that (i) every pair of durations of steps that are carried out during the time interval (s, t) has the same conditional distribution, given that the time interval of the kth step contains s and that of the lth step contains t, and (ii) every pair consisting of the duration of the step performed at time s and the duration of a step carried out in (s, t) has the same conditional distribution, given the same condition as in (i).

Fact (c) is valid without further conditions, as is most easily seen by considering the random walk generated by $\xi_i' = \varepsilon 1_{\{\xi_i \geq \varepsilon\}}$, where $\varepsilon > 0$ is chosen such that $P(\xi_i \geq \varepsilon) = \delta > 0$. The corresponding $\tau'(t)$ satisfies $\tau'(t) \geq \tau(t)$, because $\xi_i' \leq \xi_i$, and has a negative binomial distribution with parameters δ and $[t/\varepsilon] + 1$. Thus, $E(\tau'(t)^a)$ is easily calculated and estimated to give (c).

Now we proceed with the formulation and proof of the four announced lemmas.

Lemma 1. The following holds

$$\limsup_{n \to \infty} E(\|S_{n,\beta} - S_{n,\alpha}\|^2) \leq K(\beta - \alpha) \tag{2.6}$$

for all $\alpha, \beta \geq 0$, $\alpha \leq \beta$.

Proof. By our independence assumptions on $(\xi_i)_{i \geq 1}$ and $(M_i)_{i \geq 1}$ we have

$$\begin{aligned}
&E(\|\hat{X}(t) - \hat{X}(s)\|^2) \\
&= \sum_{p=0}^{\infty} \sum_{q=p+1}^{\infty} E\Bigg(1_{\{\tau(s)=p,\tau(t)=q\}} \Bigg[\sum_{i=p+1}^{q} \xi_i^2 \\
&\quad + 2 \sum_{i=p+1}^{q-1} \sum_{j=i+1}^{q} \xi_i \xi_j E(\langle P_{i-1} e, E(M)^{j-i} P_{i-1} e \rangle)\Bigg]\Bigg) \\
&= \sum_{p=0}^{\infty} \sum_{q=p+1}^{\infty} E\Bigg(1_{\{\tau(s)=p,\tau(t)=q\}} \Bigg[\xi_{p+1}^2 + (q-p-1)\xi_q^2 \\
&\quad + 2\xi_{p+1}\xi_q E(\langle P_p e, C_{q-p-1} P_p e \rangle) \\
&\quad + 2\xi_{q-1}\xi_q \sum_{i=p+2}^{q-1} E(\langle P_{i-1} e, C_{q-i} P_{i-1} e \rangle)\Bigg]\Bigg)
\end{aligned} \tag{2.7}$$

where

$$C_k = [E(M) - E(M)^{k+1}][\mathrm{Id} - E(M)]^{-1} \tag{2.8}$$

By $\|C_k\| \leqslant 2 \| \|[\mathrm{Id} - E(M)]^{-1}\|$ and the above results on $\xi_{\tau(t)}$ and $\xi_{\tau(t)\pm 1}$ we obtain

$$\begin{aligned}
&E(\|\hat{X}(t) - \hat{X}(s)\|^2) \\
&\quad \leqslant E(\xi^2_{\tau(s)+1} + [\tau(t) - \tau(s) - 1]\,\xi^2_{\tau(t)} \\
&\qquad + 4\,\|[\mathrm{Id} - E(M)]^{-1}\|\,\{\xi_{\tau(s)+1}\xi_{\tau(t)} \\
&\qquad + [\tau(t) - \tau(s) - 1]^+\,\xi_{\tau(t)-1}\xi_{\tau(t)}\}) \\
&\quad \leqslant E(\xi^2_{\tau(s)+1}) + E(|\tau(t) - \tau(s)|^a)^{1/a}\,E(\xi^{2b}_{\tau(t)})^{1/2b} \\
&\qquad + 4\,\|[\mathrm{Id} - E(M)]^{-1}\|\,[E(\xi^2_{\tau(s)+1})^{1/2} E(\xi^2_{\tau(t)})^{1/2} \\
&\qquad + E(|\tau(t) - \tau(s)|^a)^{1/a}\,E(\xi^{2b}_{\tau(t)-1})^{1/2b}\,E(\xi^{2b}_{\tau(t)})^{1/2b}] \\
&\quad \leqslant K_1 + K_2 E(|\tau(t) - \tau(s)|^a)^{1/a} \qquad (2.9)
\end{aligned}$$

for some constants K_1, $K_2 \geqslant 0$. Here $a > 1$ is a positive integer, and $b > 1$ satisfies $a^{-1} + b^{-1} = 1$ and $2b < 2 + \varepsilon$. We have used the Hölder and the Cauchy–Schwarz inequalities and the fact that $\sup_{t \geqslant 0} E(\xi^{2b}_{\tau(t)+i}) < \infty$ for $i = -1, 0, 1$.

Now let $s_n = T[\alpha n]/n$ and $t_n = T[\beta n]/n$. It follows from (2.5) and (2.12) that

$$\begin{aligned}
&\limsup_{n \to \infty} E(\|S_{n,\beta} - S_{n,\alpha}\|^2) \\
&\quad = \limsup_{n \to \infty} a_n^{-1} E(\|\hat{X}(a_n t_n) - \hat{X}(a_n s_n)\|^2) \\
&\quad \leqslant \limsup_{n \to \infty} K_2 a_n^{-1} E(|\tau(a_n t_n) - \tau(a_n t_n)|^a)^{1/a} \\
&\quad \leqslant \limsup_{n \to \infty} K_2 a_n^{-1} E(|1 + \tau(a_n(t_n - s_n))|^a)^{1/a} \\
&\quad \leqslant \limsup_{n \to \infty} K_2 a_n^{-1} \left[\sum_{i=0}^{a} \binom{a}{i} E(\tau(a_n(t_n - s_n))^i) \right]^{1/a} \\
&\quad \leqslant \limsup_{n \to \infty} K_3 E(|\tau(a_n(t_n - s_n))/a_n|^a)^{1/a} \\
&\quad \leqslant \limsup_{n \to \infty} K_4(t_n - s_n) = K_4 T(\beta - \alpha) \qquad (2.10)
\end{aligned}$$

Lemma 2. We have

$$E(\|S_{n,\alpha+\Delta} - S_{n,\alpha} \mid S_{n,\alpha})\|) = O(a_n^{-1/2}) \qquad \text{as} \quad n \to \infty \qquad (2.11)$$

for all α, $\Delta \geqslant 0$.

Proof. We introduce the σ-algebra $\mathscr{A}_{s+}$, which is generated by $\xi_1,\ldots,\xi_{\tau(t)+1}$, $M_1,\ldots,M_{\tau(t)-1}$. Various properties of conditional expectations will be tacitly used in the following. Let $A(p,q)=\{\tau(s)=p, \tau(t)=q\}$. As above, we can write

$$\begin{aligned}
&E(\hat{X}(t)-\hat{X}(s)\mid \mathscr{A}_{s+})\\
&\quad=E\left(\sum_{i=\tau(s)+1}^{\tau(t)} \xi_i P_{i-1} e \mid \mathscr{A}_{s+}\right)\\
&\quad=E\left(\sum_{p=0}^{\infty}\sum_{q=p+1}^{\infty} 1_{A(p,q)} \sum_{i=p+1}^{q} \xi_i P_{i-1} e\mid \mathscr{A}_{s+}\right)\\
&\quad=E\left(\sum_{p=0}^{\infty}\sum_{q=p+1}^{\infty} 1_{A(p,q)}\left[\xi_{p+1}P_p e+\xi_q \sum_{i=p+2}^{q} E(M)^{i-p} P_{p-1} e\right]\middle| \mathscr{A}_{s+}\right)\\
&\quad=E(1_{\{\tau(s)<\tau(t)\}}\xi_{\tau(s)+1}P_{\tau(s)}e+1_{\{\tau(s)+1<\tau(t)\}}\xi_{\tau(t)}\\
&\qquad\times[E(M)^2-E(M)^{\tau(t)-\tau(s)+1}][\mathrm{Id}-E(M)]^{-1}P_{\tau(s)-1}e\mid \mathscr{A}_{s+}) \qquad (2.12)
\end{aligned}$$

Since $\hat{X}(s)$ is a function of $\xi_1,\ldots,\xi_{\tau(s)}$, $M_1,\ldots,M_{\tau(s)-1}$, $\hat{X}(s)$ is $\mathscr{A}_{s+}$-measurable. By this remark, the simple inequality $\|E(U\mid\mathscr{A})\|\leqslant 2^{1/2}E(\|U\|\mid\mathscr{A})$, and (2.12), we obtain

$$\begin{aligned}
&E(\|E(\hat{X}(t)-\hat{X}(s)\mid\hat{X}(s))\|)\\
&\quad=E(\|E(E(\hat{X}(t)-\hat{X}(s)\mid\mathscr{A}_{s+})\mid\hat{X}(s))\|)\\
&\quad\leqslant E(2^{1/2}E(\|E(\hat{X}(t)-\hat{X}(s)\mid\mathscr{A}_{s+})\|\mid\hat{X}(s)))\\
&\quad=2^{1/2}E(\|E(\hat{X}(t)-\hat{X}(s)\mid\mathscr{A}_{s+})\|)\\
&\quad\leqslant 2^{3/2}\|[\mathrm{Id}-E(M)]^{-1}|\,E(\xi_{\tau(s)+1}+\xi_{\tau(t)})\\
&\quad\leqslant K \qquad (2.13)
\end{aligned}$$

where K is independent of s and t. Setting $s_n=[\alpha n]T/n$ and $t_n=[(\alpha+\Delta)n]T/n$, we get

$$E(\|E(\hat{X}_n(t_n)-\hat{X}_n(s_n)\mid\hat{X}_n(s_n))\|)\leqslant Ka_n^{-1/2} \qquad (2.14)$$

as claimed.

The most difficult part of this paper is the derivation of the following lemma, which contains the introduction of the asymptotic covariance matrix.

Lemma 3. We have

$$\limsup_{n\to\infty} E\left(\left\| E((S_{n,\alpha+\Delta}-S_{n,\alpha})(S_{n,\alpha+\Delta}-S_{n,\alpha})' \mid S_{n,\alpha}) - \frac{\Delta\alpha T}{2}\left[\frac{\mu_2}{\mu}\,\mathrm{Id}+\mu(N+N')\right]\right\|\right)=0 \tag{2.15}$$

for all α, $\Delta \geqslant 0$.

Proof. Let $s \leqslant t$ and

$$U=E([\hat{X}(t)-\hat{X}(s)][\hat{X}(t)-\hat{X}(s)]' \mid \mathscr{A}_{s+})$$

Since $\hat{X}(s)$ is $\mathscr{A}_{s+}$-measurable,

$$E([\hat{X}(t)-\hat{X}(s)][\hat{X}(t)-\hat{X}(s)]' \mid \hat{X}(s))=E(U \mid \hat{X}(s)) \tag{2.16}$$

Let again $A(p,q)=\{\tau(s)=p, \tau(t)=q\}$. Using the same partition technique as in the proof of the former lemmas, we can write

$$\begin{aligned}
U&=E\left(\sum_{i=\tau(s)+1}^{\tau(t)}\ \sum_{j=\tau(s)+1}^{\tau(t)} \xi_i\xi_j P_{i-1}e(P_{j-1}e)' \mid \mathscr{A}_{s+}\right)\\
&=\sum_{p=0}^{\infty}\ \sum_{q=p+1}^{\infty}\Big\{E(1_{A(p,q)}\xi_{p+1}^2 P_p e(P_p e)' \mid \mathscr{A}_{s+})\\
&\quad+\sum_{i=p+2}^{q} E(1_{A(p,q)}\,\xi_q^2 P_{i-1}e(P_{i-1}e)' \mid \mathscr{A}_{s+})\\
&\quad+\sum_{i=p+2}^{q} E(1_{A(p,q)}\xi_{p+1}\xi_q[P_{i-1}e(P_pe)'+P_pe(P_{i-1}e)'] \mid \mathscr{A}_{s+})\\
&\quad+\sum_{i=p+2}^{q-1}\ \sum_{j=i+1}^{q} E(1_{A(p,q)}\xi_{q-1}\xi_q[P_{i-1}e(P_{j-1}e)'\\
&\quad+P_{j-1}e(P_{i-1}e)'] \mid \mathscr{A}_{s+})\Big\}\\
&=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}
\end{aligned} \tag{2.17}$$

The different sums in the braces result from distinguishing the cases (a) $i=j=p+1$, (b) $i=j=p+2,\ldots,q$, (c) $i=p+2,\ldots,q$ and $j=p+1$ or vice versa, (d) $i, j \in \{p+2,\ldots,q\}$ and $i \neq j$. Now we introduce some auxiliary quantities of use for rewriting the sums I–IV in a more convenient form. Let $Y=P_{\tau(s)-1}e$ and let A, A_1, A_2,... be independent random matrices, which are also independent of all M_i and ξ_j and all have the same distribution as M. Further, we define

$$R_{n,k} = \frac{1}{\tau(t)-\tau(s)} \sum_{m=2}^{\tau(t)-\tau(s)-k} E(A_m \cdots A_1 YY'A_1' \cdots A_m' \mid Y), \qquad k=0,1 \tag{2.18}$$

where $0/0 := 0$, a case occurring iff $\tau(s)=\tau(t)$. Note that $R_{n,k}=0$ if $\tau(t)-\tau(s) \leqslant 1-k$. It follows from our independence assumptions and some algebra that we can write

$$\mathrm{I} = E(\xi^2_{\tau(s)+1} E(AYY'A' \mid Y) \mid \mathscr{A}_{s+}) \tag{2.19}$$

$$\mathrm{II} = E([\tau(t)-\tau(s)]\, \xi^2_{\tau(t)} R_{n,0} \mid \mathscr{A}_{s+}) \tag{2.20}$$

$$\begin{aligned}\mathrm{III} = E(1_{\{\tau(s)+1<\tau(t)\}}\, &\xi_{\tau(s)+1}\xi_{\tau(t)}\{[E(M)-E(M)^{\tau(t)-\tau(s)}] \\ &\times [\mathrm{Id}-E(M)]^{-1} E(AYY'A' \mid Y) \\ &+ E(AYY'A' \mid Y)[E(M')-E(M')^{\tau(t)-\tau(s)}][\mathrm{Id}-E(M')]^{-1}\} \mid \mathscr{A}_{s+})\end{aligned} \tag{2.21}$$

$$\begin{aligned}\mathrm{IV} = E(&\xi_{\tau(t)-1}\xi_{\tau(t)}[\tau(t)-\tau(s)]\{R_{n,1} E(M')[\mathrm{Id}-E(M')]^{-1} \\ &+ E(M)[\mathrm{Id}-E(M)]^{-1} R_{n,1}\} \mid \mathscr{A}_{s+}) \\ &- E(\xi_{\tau(t)-1}\xi_{\tau(t)} \sum_{m=2}^{\tau(t)-\tau(s)-1} [E(A_m \cdots A_1 YY'A_1' \cdots A_m' \mid Y) \\ &\times E(M')^{\tau(t)-\tau(s)-m+1}[\mathrm{Id}-E(M')]^{-1} \\ &+ E(M)^{\tau(t)-\tau(s)-m+1}[\mathrm{Id}-E(M)]^{-1} \\ &\times E(A_m \cdots A_1 YY'A_1' \cdots A_m' \mid Y)] \mid \mathscr{A}_{s+}) \\ = \mathrm{IV}^{(1)} &- \mathrm{IV}^{(2)}\end{aligned} \tag{2.22}$$

Now let $s_n = T[\alpha n]/n$, $t_n = T[\alpha + \Delta]/n$, and

$$U_n = E([\hat{X}_n(t_n) - \hat{X}_n(s_n)][\hat{X}_n(t_n) - \hat{X}_n(s_n)]' \mid \mathscr{A}_{s_n+}) \tag{2.23}$$

To prove the lemma, we must show that

$$E\left(\left\| E\left(U_n - \frac{\Delta\alpha T}{2}\left[\frac{\mu_2}{\mu}\,\mathrm{Id} + \mu(N+N')\right] \middle| \hat{X}_n(s_n)\right)\right\|\right) \to 0 \qquad \text{as} \quad n \to \infty \tag{2.24}$$

It clearly suffices to prove

$$E\left(\left\| U_n - \frac{\Delta\alpha T}{2}\left[\frac{\mu_2}{\mu}\,\mathrm{Id} + \mu(N+N')\right]\right\|\right) \to 0 \qquad \text{as} \quad n\to\infty \tag{2.25}$$

Relation (2.25) will be derived by considering the following crucial decomposition of the integrand in (2.25):

$$\begin{aligned} U_n &- \frac{\Delta\alpha T}{2}\left[\frac{\mu_2}{\mu}\,\mathrm{Id} + \mu(N+N')\right] \\ &= E\left(\left\{a_n^{-1}[\tau(a_n t_n) - \tau(a_n s_n)] - \frac{\Delta\alpha T}{\mu}\right\}[\xi^2_{\tau(a_n t_n)} R_{n,0}\right. \\ &\quad \left. + \xi_{\tau(a_n t_n)}\xi_{\tau(a_n t_n)-1}(R_{n,1}N' + NR_{n,1})] \mid \mathscr{A}_{a_n s_n +}\right) \\ &\quad + \frac{\Delta\alpha T}{\mu} E\left(\xi^2_{\tau(a_n t_n)}\left(R_{n,0} - \frac{1}{2}\,\mathrm{Id}\right) + \left(R_{n,1} - \frac{1}{2}\,\mathrm{Id}\right) N'\right. \\ &\quad \left. + N\left(R_{n,1} - \frac{1}{2}\,\mathrm{Id}\right) \xi^2_{\tau(a_n t_n)}\xi^2_{\tau(a_n t_n)-1} \mid \mathscr{A}_{a_n s_n +}\right) \\ &\quad + \frac{\Delta\alpha T}{\mu}[E(\xi^2_{\tau(a_n t_n)} \mid \mathscr{A}_{a_n s_n +}) - \mu_2]\,\frac{1}{2}\,\mathrm{Id} \\ &\quad + \frac{\Delta\alpha T}{2\mu}[E(\xi_{\tau(a_n t_n)}\xi_{\tau(a_n t_n)-1} \mid \mathscr{A}_{a_n s_n +}) - \mu^2](N+N') \\ &\quad + a_n^{-1}(\mathrm{I}_n + \mathrm{III}_n - \mathrm{IV}_n^{(2)}) \end{aligned} \tag{2.26}$$

where I_n, III_n, and $\mathrm{IV}_n^{(2)}$ are the quantities corresponding to I, III, and $\mathrm{IV}^{(2)}$ with t and s replaced by $a_n t_n$ and $a_n s_n$.

Using (2.17) and (2.19)–(2.22) for U_n instead of U, this decomposition is easily checked. Now the proof of (2.25) is carried out by successively estimating the terms on the right-hand side of (2.26) back to front. We start by noting that

$$E(\|a_n^{-1} I_n\|) \leqslant a_n^{-1} K E(\xi^2_{\tau(a_n s_n)+1}) = O(a_n^{-1}) \tag{2.27}$$

and, by the Cauchy–Schwarz inequality,

$$E(\|a_n^{-1}\cdot \mathrm{III}_n\|) \leqslant a_n^{-1} K E(\xi^2_{\tau(a_n s_n)+1})^{1/2} E(\xi^2_{\tau(a_n t_n)}) = O(a_n^{-1}) \tag{2.28}$$

For estimating $E(\|a_n^{-1}\cdot \mathrm{IV}_n^{(2)}\|)$ we choose a sufficiently large $a \in \mathbb{N}$ and a corresponding $b>1$ such that $a^{-1}+b^{-1}=1$. As in the proof of Lemma 1, repeated use of the Hölder inequality yields

$$\begin{aligned}
&E(\|a_n^{-1}\cdot \mathrm{IV}_n^{(2)}\|)\\
&\quad\leqslant KE(\xi^{2b}_{\tau(a_nt_n)-1})^{1/2b}\,E(\xi^{2b}_{\tau(a_nt_n)})^{1/2b}\\
&\qquad\times a_n^{-1}E(|\tau(a_nt_n)-\tau(a_ns_n)|^{2a})^{1/2a}\\
&\qquad\times E\Bigg(\Bigg\|[\tau(a_nt_n)-\tau(a_ns_n)]^{-1}\\
&\qquad\times\sum_{m=2}^{\tau(a_nt_n)-\tau(a_ns_n)-1}[E(A_m\cdots A_1Y_nY_n'A_1'\cdots A_m'\mid Y_n)\\
&\qquad\times E(M')^{\tau(a_nt_n)-\tau(a_ns_n)-m+1}\\
&\qquad+E(M)^{\tau(a_nt_n)-\tau(a_ns_n)-m+1}E(A_m\cdots A_1Y_nY_n'A_1'\cdots A_m'\mid Y)]\Bigg\|^{2a}\Bigg)^{1/2a}
\end{aligned}\tag{2.29}$$

where now $Y_n=P_{\tau(a_ns_n)-1}e$. By our assumption $E(\xi_1^{3+\varepsilon})<\infty$, we again conclude that

$$\sup_{n\geqslant 1}E(\xi^{2b}_{\tau(a_nt_n)-1})<\infty,\qquad \sup_{n\geqslant 1}E(\xi^{2b}_{\tau(a_nt_n)})<\infty$$

As in (2.10), it is seen that

$$\sup_{n\geqslant 1}a_n^{-1}E(|\tau(a_nt_n)-\tau(a_ns_n)|^{2a})^{1/2a}<\infty\tag{2.30}$$

Thus, the first factors on the right-hand side of (2.29) are bounded. Concerning the main term in (2.29), we now show that

$$E(\eta_n^{2a})\to 0\qquad\text{as}\quad n\to\infty\tag{2.31}$$

where we have set

$$\eta_n=\left\|n^{-1}\sum_{m=1}^{n}E(M)^{n-m+1}E(A_m\cdots A_1Y_nY_n'A_1'\cdots A_m'\mid Y_n)\right\|\tag{2.32}$$

Relations (2.30) and (2.31) entail $E(a_n^{-1}\|\mathrm{IV}_n^{(2)}\|)\to 0$, because $\tau(a_nt_n)-\tau(a_ns_n)$ is stochastically independent of η_n and converges almost surely to infinity, so that by the formula of total probability

$$E(\eta^{2a}_{\tau(a_nt_n)-\tau(a_ns_n)})=\sum_{k=0}^{\infty}E(\eta_k^{2a})\,P(\tau(a_nt_n)-\tau(a_ns_n)=k)\to 0\qquad\text{as}\quad n\to\infty$$

Hence we must prove (2.31). Note that

$$\eta_n = \left\| E\left(A_{n+1} \cdots A_1 Y_n Y_n' A_1' \cdots A_{n+1}' n^{-1} \sum_{m=1}^{n} A_{n+1} \cdots A_{m+1} \mid Y_n \right) \right\| \tag{2.33}$$

Since the integrand in (2.33) is uniformly bounded, it is sufficient to prove

$$n^{-1} \sum_{m=1}^{n} A_{n+1} \cdots A_{m+1} \xrightarrow{D} 0 \qquad \text{as} \quad n \to \infty \tag{2.34}$$

Thus, let the random matrix B be a weak accumulation point of $n^{-1}\sum_{m=1}^{n} A_{n+1} \cdots A_{m+1}$. Since

$$\begin{aligned} E\Bigg(n^{-1} \sum_{m=1}^{n} A_{n+1} \cdots A_{m+1} \left[n^{-1} \sum_{m=1}^{n} A_{n+1} \cdots A_{m+1} \right]'\Bigg) & \\ = n^{-1} \,\mathrm{Id} + n^{-2} \sum_{p=1}^{n-1} E(A_{n+1} \cdots A_{p+1} [E(A) - E(A)^{n-p+1}] & \\ \times [\mathrm{Id} - E(A)]^{-1} A_{p+1}' \cdots A_{n+1}') & \\ + n^{-2} \sum_{p=1}^{n-1} E(A_{n+1} \cdots A_{p+1} [E(A') - E(A')^{n-p+1}] & \\ \times [\mathrm{Id} - E(A')]^{-1} A_{p+1}' \cdots A_{n+1}') & \\ \to 0 \qquad \text{as} \quad n \to \infty & \end{aligned} \tag{2.35}$$

we immediately have $E(BB') = 0$, so that $B = 0$ almost surely. If follows that

$$E(\|a_n^{-1} \cdot \mathrm{IV}_n^{(2)}\|) \to 0 \qquad \text{as} \quad n \to \infty \tag{2.36}$$

Going to the next term in (2.26), we must show that

$$E(|E(\xi^2_{\tau(a_n t_n)} \mid \mathscr{A}_{a_n s_n +}) - \mu_2|) \to 0, \qquad \text{as} \quad n \to \infty \tag{2.37}$$

Here we have to use the elementary renewal theorem $\lim_{t \to \infty} \tau(t)/t = 1/\mu$ almost surely (see, e.g., ref. 17, p. 127).

Let $t' = a_n t_n$, $s' = a_n s_n$, and

$$N_\varepsilon = \{ p \in \mathbb{N} \mid |p/s' - 1/\mu| < \varepsilon \}$$

$$\tilde{N}_\varepsilon = \{ q \in \mathbb{N} \mid |q/t' - 1/\mu| < \varepsilon \}$$

where $\varepsilon > 0$ is so small that $\sup N_\varepsilon + 1 < \inf \tilde{N}_\varepsilon$. Then it is clear from the elementary renewal theorem that, with probability close to 1, $\tau(t')$ and $\tau(s')$ will belong to $\tilde{N}_\varepsilon$ and N_ε, respectively. Therefore, we decompose $E(\xi^2_{\tau(t')} \mid A_{s'+})$ as follows:

$$\begin{aligned}
E(\xi^2_{\tau(t')} \mid \mathscr{A}_{s'+}) &= \sum_{p \in N_\varepsilon, q \in \tilde{N}_\varepsilon} E(1_{\{\tau(s')=p, \tau(t')=q\}} \xi^2_q \mid \xi_1, \dots, \xi_{p+1}) + R \\
&= \sum_{p \in N_\varepsilon} E(1_{\{\tau(s')=p\}} \xi^2_{p+2} \mid \xi_1, \dots, \xi_{p+1}) \\
&\quad - \sum_{p \in N_\varepsilon, q \in \mathbb{N} \setminus \tilde{N}_\varepsilon} E(1_{\{\tau(s')=p, \tau(t')=q\}} \xi^2_{p+2} \mid \xi_1, \dots, \xi_{p+1}) + R \qquad (2.38)
\end{aligned}$$

where the remainder term R is given by

$$R = E(1_{\{\tau(s') \notin N_\varepsilon \text{ or } \tau(t') \notin \tilde{N}_\varepsilon\}} \xi^2_{\tau(t')} \mid \mathscr{A}_{s'+}) \qquad (2.39)$$

Since the first sum on the right-hand side is equal to

$$\sum_{p \in N_\varepsilon} 1_{\{\tau(s')=p\}} E(\xi^2_1) = \mu_2 P(\tau(s') \in N_\varepsilon) \qquad (2.40)$$

it follows immediately from (2.38) that

$$\begin{aligned}
&E(|E(\xi_{\tau(t')} \mid \mathscr{A}_{s'+}) - \mu_2|) \\
&\quad \leqslant \mu_2(1 - P(\tau(s') \in N_\varepsilon)) + E(\xi^2_{\tau(t')}[2 \cdot 1_{\{\tau(t') \notin \tilde{N}_\varepsilon\}} + 1_{\{\tau(s') \notin N_\varepsilon\}}]) \\
&\quad \leqslant \mu_2 P(\tau(s') \notin N_\varepsilon) + [2P(\tau(t') \notin \tilde{N}_\varepsilon)^{1/a} + P(\tau(s') \notin N_\varepsilon)^{1/a}] \, E(\xi^{2b}_{\tau(t')})^{1/2b} \\
&\quad \to 0 \qquad (2.41)
\end{aligned}$$

where $a, b > 1$ are chosen such that $a^{-1} + b^{-1} = 1$ and $2b < 2 + \varepsilon$. The convergence to zero follows from $\lim_{t \to \infty} \tau(t)/t = 1/\mu$ almost surely.

In the same way it is proved that

$$E(|E(\xi^2_{\tau(a_n t_n)} \xi_{\tau(a_n t_n)-1} \mid \mathscr{A}_{a_n s_n +}) - \mu^2|) \to 0 \qquad \text{as} \quad n \to \infty \qquad (2.42)$$

For the two first terms on the right-hand side of (2.26) we have to show that for sufficiently large integers $a > 1$

$$\lim_{n \to \infty} E\left(\left\| R_{n,k} - \frac{1}{2} \mathrm{Id} \right\|^a \right) = 0, \qquad k = 0, 1 \quad (2.43)$$

$$\lim_{n \to \infty} E\left(\left| a_n^{-1}[\tau(a_n t_n) - \tau(a_n s_n)] - \frac{\Delta \alpha T}{\mu} \right|^a \right) = 0 \qquad (2.44)$$

To derive (2.43), it is enough to prove $R_{n,k} \xrightarrow{D} \frac{1}{2}\,\mathrm{Id}$ as $n \to \infty$, since the sequences $R_{n,k}$ are uniformly bounded. For this it suffices to show that

$$B_n := n^{-1} \sum_{m=1}^{n} E(A_m \cdots A_1 Y_n Y_n' A_m' \cdots A_1' \mid Y_n) \xrightarrow{D} \frac{1}{2}\,\mathrm{Id} \tag{2.45}$$

i.e., each weak accumulation point B of B_n is equal to $\frac{1}{2}\,\mathrm{Id}$ almost surely. We may assume $B_n \xrightarrow{D} B$. Obviously we have $E(AB_nA' \mid B_n) \to E(ABA' \mid B)$, because A is independent of B_n and B. On the other hand,

$$\begin{aligned} B_n &= n^{-1} \sum_{m=1}^{n} E(AE(A_{m-1} \cdots A_1 Y_n Y_n' A_1' \cdots A_{m-1}' \mid Y_n) A' \mid Y_n) \\ &= n^{-1}[E(AY_n Y_n' A' \mid Y_n) - E(A_{n+1} \cdots A_1 Y_n Y_n' A_1' \cdots A_{n+1}' \mid Y_n)] \\ &\quad + E(AB_nA' \mid Y_n) \end{aligned} \tag{2.46}$$

The first term on the right-hand side of (2.46), say $n^{-1}C_n$, has a norm $\leqslant K/n$ for some constant K. Since B_n is a function of Y_n, we can conclude that

$$\begin{aligned} \|B_n - E(AB_nA' \mid B_n)\| &= \|E(B_n - E(AB_nA' \mid Y_n) \mid B_n)\| \\ &= \|E(n^{-1}C_n \mid B_n)\| \leqslant K/n \to 0 \end{aligned} \tag{2.47}$$

Thus, B satisfies $B = E(ABA' \mid B)$. We conclude from this equation that $B = \frac{1}{2}\,\mathrm{Id}$ almost surely. Clearly we may assume $B \equiv \text{const}$, so that $B = E(ABA')$. Indeed, suppose that for every constant (nonrandom) symmetric matrix b of trace 1 the equation $b = E(AbA')$ implies $b = \frac{1}{2}\,\mathrm{Id}$. Then it follows from the relation

$$b = E(ABA' \mid B = b) = E(AbA')$$

which holds for almost all possible values of the random matrix B, that $B = \frac{1}{2}\,\mathrm{Id}$ almost surely.

Now let $B \equiv \text{const}$. Since B is symmetric, there are an orthogonal matrix U and a diagonal matrix D such that $B = UDU'$. Then $D = E(\tilde{A} D \tilde{A}')$, where $\tilde{A} = U'AU$. Let

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \qquad \tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \tag{2.48}$$

Assume $d_1 \neq d_2$. The equations

$$d_1 = d_1 E(\tilde{A}_{11}^2) + d_2 E(\tilde{A}_{12}^2), \qquad d_2 = d_1 E(\tilde{A}_{21}^2) + d_2 E(\tilde{A}_{22}^2) \tag{2.49}$$

entail $E(\tilde{A}_{11}^2) = E(\tilde{A}_{22}^2) = 1$, so that $\tilde{A}$ is concentrated on the four matrices

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$

Thus, with probability 1, A maps the line $\{U'(\begin{smallmatrix} x \\ 0 \end{smallmatrix}) \mid x \in \mathbb{R}\}$ to itself, a case we have excluded by (1.3). Hence, it follows that $d_1 = d_2$, and since B has trace 1, $d_1 = d_2 = 1/2$. Thus, (2.43) is proved.

To show (2.44), note that $\tau(a_n t_n)/a_n \to T(\alpha + \Delta)/\mu$ and $\tau(a_n s_n)/a_n \to T\alpha/\mu$ almost surely. Thus, it suffices to prove that $|a_n^{-1}[\tau(a_n t_n) - \tau(a_n s_n)]|$ is uniformly integrable. But this is a consequence of

$$\sup_{n \geqslant 1} E(|a_n^{-1}[\tau(a_n t_n) - \tau(a_n s_n)]|^{a+1})$$
$$\leqslant \sup_{n \geqslant 1} a_n^{-a-1} E(|1 + \tau(a_n(t_n - s_n))|^{a+1}) < \infty$$

where the last inequality follows as in the proof of Lemma 1.

We have now shown that for every term on the right-hand side of (2.26) the expected value of its norm tends to zero. This completes the proof of Lemma 3.

Lemma 4. We have

$$\lim_{\Delta \to 0+} \frac{1}{\Delta} \limsup_{n \to \infty} \int_{\{\|S_{n,\alpha+\Delta} - S_{n,\alpha}\| \geqslant \varepsilon\}} \|S_{n,\alpha+\Delta} - S_{n,\alpha}\|^2 \, dP = 0 \tag{2.50}$$

for all $\alpha \geqslant 0$ and all $\varepsilon > 0$.

Proof. Since

$$\int_{\{\|S_{n,\alpha+\Delta} - S_{n,\alpha}\|^2 \geqslant \varepsilon\}} \|S_{n,\alpha+\Delta} - S_{n,\alpha}\|^2 \, dP$$
$$= \int_{\varepsilon}^{\infty} P(\|S_{n,\alpha+\Delta} - S_{n,\alpha}\| \geqslant \sqrt{t}) \, dt + \varepsilon^2 P(\|S_{n,\alpha+\Delta} - S_{n,\alpha}\| \geqslant \varepsilon) \tag{2.51}$$

by an integration by parts and

$$S_{n,\alpha+\Delta} - S_{n,\alpha} = \hat{X}_n(T[(\alpha + \Delta)n]/n) - \hat{X}_n(T[\alpha n]/n) \tag{2.52}$$

it suffices to prove the following: There are constants $K > 0$ and $\beta > 1$ such that for all $\delta_0 > 0$ there exists an $n_0 = n_0(\delta_0) \in \mathbb{N}$ satisfying

$$P(\|\hat{X}_n(t) - \hat{X}_n(s)\| \geqslant \delta) \leqslant K(t - s)^{\beta}/\delta^{2\beta} \tag{2.53}$$

for all $t \geqslant s \geqslant 0$, $\delta \geqslant \delta_0$, and $n \geqslant n_0$. For, if the existence of K and β is established, the right-hand side of (2.51) can be estimated from above by

$$\int_\varepsilon^\infty K\left\{\frac{T}{n}([(\alpha+\Delta)n]-[\alpha n])\right\}^\beta t^{-\beta}\,dt+\varepsilon^2\frac{\{(T/n)([(\alpha+\Delta)n]-[\alpha n])\}^\beta}{\varepsilon^{2\beta}}$$
$$\leqslant \tilde{K}(\Delta^\beta\varepsilon^{1-\beta}+\Delta^\beta\varepsilon^{2-2\beta}) \tag{2.54}$$

if n is large enough to ensure the applicability of (2.53) to both terms on the right-hand side of (2.51), i.e., $n \geqslant \max[n_0(\varepsilon), n_0(\sqrt{\varepsilon})]$. Further, the constant $\tilde{K}$ does not depend on ε and Δ. Hence,

$$\frac{1}{\Delta}\limsup_{n\to\infty}\int_{\{\|S_{n,\alpha+\Delta}-S_{n,\alpha}\|^2\geqslant\varepsilon\}}\|S_{n,\alpha+\Delta}-S_{n,\alpha}\|^2\,dP$$
$$\leqslant \tilde{K}(\varepsilon^{1-\beta}+\varepsilon^{2-2\beta})\,\Delta^{\beta-1} \tag{2.55}$$

for each fixed $\varepsilon>0$, and since $\beta>1$, the right-hand side of (2.55) tends to zero as $\Delta\to 0+$. Thus, it remains to show (2.53).

Without restriction of generality, assume that $\sqrt{a_n}=m_n$ is an integer. Let $s_j=s+(t-s)j/m_n$, $j=0,1,..., m_n$. Obviously $\|\hat{X}_n(t)-\hat{X}_n(s)\|\geqslant\delta$ entails $\max_j\|\hat{X}_n(s_j)-\hat{X}_n(s_{j-1})\|\geqslant\delta/2$ or

$$\max_j\min[\|\hat{X}_n(s_j)-\hat{X}_n(s)\|, \|\hat{X}_n(t)-\hat{X}_n(s_j)\|]\geqslant\delta/4$$

Therefore,

$$P(\|\hat{X}_n(t)-\hat{X}_n(s)\|\geqslant\delta)$$
$$\leqslant P(\max_j\|\hat{X}_n(s_j)-\hat{X}_n(s_{j-1})\|\geqslant\delta/2)$$
$$+P(\max_j\min[\|\hat{X}_n(s_j)-\hat{X}_n(s)\|, \|\hat{X}_n(t)-\hat{X}_n(s_j)\|]\geqslant\delta/4) \tag{2.56}$$

Now choose $\delta_0>0$ and let $\delta\geqslant\delta_0$ be arbitrary. The first probability on the right-hand side of (2.56) is equal to zero for $n\geqslant n_0(\delta_0)$. This follows from

$$\|\hat{X}_n(s_j)-\hat{X}_n(s_{j-1})\|\leqslant s_j-s_{j-1}=(t-s)/m_n$$

One has to choose $n_0(\delta_0)$ such that $m_n>2(t-s)/\delta_0$ for $n\geqslant n_0(\delta_0)$.

To estimate the second probability on the right-hand side of (2.56), we use a theorem of Billingsley (ref. 16, p. 89) in the following form: If $Y_1, Y_2,...$ is an arbitrary sequence of random variables, $S_j=Y_1+\cdots+Y_j$, and if for some $\alpha>0$ and $\beta>1$

$$P(\|S_j-S_i\|\geqslant\delta, \|S_k-S_j\|\geqslant\delta)\leqslant\delta^{-2\beta}(\alpha(k-i)(t-s))^\beta$$
$$\text{for all}\quad 0\leqslant i\leqslant j\leqslant k\leqslant m, \quad \delta>0 \tag{2.57}$$

then one can conclude that

$$P(\max_{0 \leqslant j \leqslant m} \min[\|S_j\|, \|S_m - S_j\|] \geqslant \delta) \leqslant K\delta^{-2\beta}(m\alpha(t-s))^{\beta} \qquad (2.58)$$

Let $S_j^{(n)} = \hat{X}_n(s_j) - \hat{X}_n(s)$. To check condition (2.57) for these sums, note that at some time instant t', the future process $(\hat{X}_n(t))_{t \geqslant t'}$ depends on the past only through the direction of the motion at time t' and the residual time up to the first jump following t', thereafter looking like the original process starting at the origin. Formally, this can be expressed as follows: For $t'' \geqslant t'$

$$\hat{X}_n(t'') = \hat{X}_n(t') + Y_n\left(\left(t'' - \sum_{l=1}^{\tau(a_n t')+1} \xi_l\right)^+, \eta_n\right) \qquad (2.59)$$

Here $\eta_n = P_{\tau(a_n t')+1} e$, and for each fixed $y \in \mathbb{R}^2$ such that $\|y\| = 1$ the process $(Y_n(t, y))_{t \geqslant 0}$ has the same distribution as $\hat{X}_n$, but with initial direction y, and $(Y_n(t, y))_{t \geqslant 0}$ is independent of $(\hat{X}_n(t))_{t \leqslant t'}$, $\xi_{\tau(a_n t')+1}$, and η_n. Now let Q_{nj} be the joint distribution of the pair

$$\left(\left(t'' - \sum_{l=1}^{\tau(a_n t')+1} \xi_l\right)^+, \eta_n\right)$$

and set $t' = s_j$, $t'' = s_k$ for arbitrary $0 \leqslant i \leqslant j \leqslant k \leqslant m_n$. Then by the formula of total probability,

$$\begin{aligned}
&P(\|S_j^{(n)} - S_i^{(n)}\| \geqslant \delta, \|S_k^{(n)} - S_j^{(n)}\| \geqslant \delta) \\
&= \int P\left(\|\hat{X}_n(s_j) - \hat{X}_n(s_i)\| \geqslant \delta, \|Y_n(u, y)\| \geqslant \delta \,\middle|\, \left(\left(s_k - \sum_{l=1}^{\tau(a_n s_j)+1} \xi_l\right)^+, \eta_n\right) = (u, y)\right) dQ_{nj}(u, y) \\
&= \int P(\|Y_n(u, y)\| \geqslant \delta)\, P\left(\|\hat{X}_n(s_j) - \hat{X}_n(s_i)\| \geqslant \delta \,\middle|\, \left(\left(s_k - \sum_{l=1}^{\tau(a_n s_j)+1} \xi_l\right)^+, \eta_n\right) = (u, y)\right) dQ_{nj}(u, y) \\
&\leqslant \sup_{0 \leqslant u \leqslant s_k - s_j} \delta^{-2} E(\|Y_n(u, \delta)\|^2)\, P(\|\hat{X}_n(s_j) - \hat{X}_n(s_i)\| \geqslant \delta) \\
&\leqslant K_1 \delta^{-2}(s_k - s_j)\, K_2 \delta^{-2}(s_j - s_i) \\
&\leqslant K\delta^{-4} m_n^{-2}[(k-i)(t-s)]^2 \qquad (2.60)
\end{aligned}$$

For the second equation we have used that $Y_n(u, y)$ is independent of $\|\hat{X}_n(s_j) - \hat{X}_n(s_i)\|$ and of the conditioning random variable. The inequalities follow from Chebyshev's inequality and Lemma 1. Thus, (2.55) holds with $\beta = 2$ and $\alpha = K^{1/2}/m_n$, so that

$$P(\max_j \min[\|S_j^{(n)}\|, \|S_{m_n}^{(n)} - S_j^{(n)}\|] \geqslant \delta) \leqslant K\delta^{-4}(t-s)^2 \tag{2.61}$$

Inserting (2.61) into (2.54) yields

$$P(\|\hat{X}_n(t) - \hat{X}_n(s)\| \geqslant \delta) \leqslant K(t-s)^2/\delta^4 \qquad \text{for all} \quad t \geqslant s \geqslant 0 \tag{2.62}$$

if $\delta \geqslant \delta_0$ and $n \geqslant n_0$ for some n_0 depending only on δ_0. Thus, (2.53) holds. The lemma is proved.

Lemmas 1–4 obviously imply the conditions 1–4 of Theorem 2, and Theorem 1 follows immediateley.

3. CONCLUSION

In this paper we have studied a two-dimensional random walk $X(t)$ with random step lengths and random directions which are generated by iterating a sequence of independent random orthogonal matrices. This type of random walk is often used to describe the motion of cells on planar surfaces. Several properties of the increments $X(t+\Delta) - X(t)$ of the process, unconditional as well as conditional on $X(t)$, have been derived. From this detailed analysis we have been able to conclude the asymptotic normality of the rescaled process $a^{-1/2}X(at)$ as $a \to \infty$ and to determine its asymptotic covariance, which turns out to depend on the first two moments of the step lengths and the matrix $E(M)[\mathrm{Id} - E(M)^{-1}] + E(M')[\mathrm{Id} - E(M')]^{-1}$ (M being one of the random changes of direction). Although the asymptotic normality seems intuitively obvious and has been conjectured by other authors, the exact proof given here requires a heavy machinery. The condition $E(\xi^{3+\varepsilon}) < \infty$, where ξ is a step length, apparently is indispensable. As already mentioned, our derivation also leads to a neat formula for the asymptotic covariance matrix. Hopefully, the methods developed in this paper will also prove useful for the analysis of more general random walks of the type considered here.

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